THE HOMOTOPY ORBIT SPECTRUM FOR PROFINITE GROUPS

DANIEL G. DAVIS[1]

Abstract. Let G be a profinite group. We define an $S[[G]]$-module to be a G-spectrum X that satisfies certain conditions, and, given an $S[[G]]$-module X, we define the homotopy orbit spectrum X_{hG}. When G is countably based and X satisfies a certain finiteness condition, we construct a homotopy orbit spectral sequence whose E_2-term is the continuous homology of G with coefficients in the graded profinite $\widehat{\mathbb{Z}}[[G]]$-module $\pi_*(X)$. Let G_n be the extended Morava stabilizer group and let E_n be the Lubin-Tate spectrum. As an application of our theory, we show that the function spectrum $F(E_n, L_{K(n)}(S^0))$ is an $S[[G_n]]$-module with an associated homotopy orbit spectral sequence.

1. Introduction

Let G be a finite group and let X be a (left) G-spectrum. Then the *homotopy orbit spectrum* X_{hG} is defined to be $\mathrm{hocolim}_G\, X$, the homotopy colimit of the G-action on X (see, for example, [16, pg. 42]). Furthermore, there is a homotopy orbit spectral sequence

$$H_p(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X_{hG}),$$

where the E_2-term is the group homology of G, with coefficients in the graded G-module $\pi_*(X)$ ([17, 5.1]). In this paper, under certain hypotheses, we extend these constructions to the case where G is a profinite group.

After making a few comments about notation, we summarize the contents of this paper. We follow the convention that all of our spectra are in Spt, the category of Bousfield-Friedlander spectra of simplicial sets. We use $(-)_{\mathtt{f}}$ to denote functorial fibrant replacement in the category of spectra; for any spectrum Z, $Z \to Z_{\mathtt{f}}$ is a weak equivalence with $Z_{\mathtt{f}}$ fibrant. Also, "holim" always denotes the version of the homotopy limit of spectra that is constructed levelwise in the category of simplicial sets, as defined in [2] and [22, 5.6].

In Section 2, for a finite group G, we use the fact that the homotopy colimit of a diagram of pointed simplicial sets is the diagonal of the simplicial replacement of the diagram, to obtain an alternative model for homotopy orbits. Then, in Section 3, we use this model to define the homotopy orbit spectrum for a profinite group G and a certain type of G-spectrum X, which we now define. We point out that the following definition was essentially first formulated by Mark Behrens.

Definition 1.1. Given a profinite group G, let $\{N_i\}$ be a collection of open normal subgroups of G such that $G \cong \lim_i G/N_i$. Then we let $S[[G]]$ be the spectrum $\mathrm{holim}_i(S[G/N_i])_{\mathtt{f}}$. Also, let $\{X_i\}$ be an inverse system of G-spectra and

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G-equivariant maps indexed over $\{i\}$, such that, for each i, the G-action on X_i factors through G/N_i (so that X_i is a G/N_i-spectrum) and X_i is a fibrant spectrum. Then we call the associated G-spectrum $X = \mathrm{holim}_i X_i$ an *$S[[G]]$-module*.

After defining the homotopy orbit spectrum X_{hG} for an $S[[G]]$-module X, we show that this construction agrees with the classical definition when G is finite. Now let us recall a definition (see [4, Section 3] for more detail).

Definition 1.2. A *discrete right G-spectrum* X is a spectrum of pointed simplicial sets X_k, for $k \geq 0$, such that each simplicial set X_k is a pointed simplicial discrete right G-set, and each bonding map $S^1 \wedge X_k \to X_{k+1}$ is G-equivariant (S^1 has trivial G-action).

In Section 4, given a discrete right G-spectrum X and any spectrum Z, we show that the function spectrum $F(X,Z)$ is an $S[[G]]$-module, so that one can form $F(X,Z)_{hG}$. Interesting examples of $F(X,Z)$ are the Brown-Comenetz dual cX of X and the Spanier-Whitehead dual DX.

Now we summarize the main result of Section 5, in which we build a homotopy orbit spectral sequence.

Recall that a topological space Y is *countably based* if there is a countable family $\mathcal{B}$ of open sets, such that each open set of Y is a union of members of $\mathcal{B}$. In particular, one can consider countably based profinite groups. Then, if G is a countably based profinite group, by [24, Proposition 4.1.3], G has a chain

$$N_0 \geq N_1 \geq N_2 \geq \cdots$$

of open normal subgroups, such that $G \cong \lim_{i\geq 0} G/N_i$. For example, a compact p-adic analytic group is a countably based profinite group. (If a topological group G is compact p-adic analytic, then G is a profinite group with an open subgroup H of finite rank, by [7, Corollary 8.34]. Then, by [7, Exercise 1, pg. 58], G has finite rank, and thus, G is finitely generated ([7, pg. 51]), so that G is countably based, by [24, pg. 55].)

Let G be a countably based profinite group, with a chain $\{N_i\}_{i\geq 0}$ of open normal subgroups, such that $G \cong \lim_i G/N_i$, and let $\{X_i\}$ be a diagram of G-spectra, such that $\mathrm{holim}_i X_i$ is an $S[[G]]$-module. Recall from [3, pg. 5] that a spectrum X is an *f-spectrum*, if, for each integer q, the abelian group $\pi_q(X)$ is finite. Let $\widehat{\mathbb{Z}}$ denote $\lim_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$. Then, if each X_i is an f-spectrum, we show that there is a homotopy orbit spectral sequence

$$E_2^{p,q} \cong H_p^c(G, \pi_q(\operatorname*{holim}_i X_i)) \Rightarrow \pi_{p+q}((\operatorname*{holim}_i X_i)_{hG}),$$

where the E_2-term is the continuous homology of G with coefficients in the graded profinite $\widehat{\mathbb{Z}}[[G]]$-module $\pi_*(\mathrm{holim}_i X_i)$.

We continue to assume that G is a countably based profinite group in Section 6, where we prove a result about the homotopy orbits of a tower of Eilenberg-Mac Lane spectra that was suggested to the author by Mark Behrens. To be precise, let $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$ be a tower of G-modules and G-equivariant maps, such that, for each i, the G-action on A_i factors through G/N_i, so that A_i is a G/N_i-module. If each abelian group A_i is finite, then we call $\{A_i\}$ a *nice tower* of G-modules. Given a nice tower $\{A_i\}$ of G-modules, we show that $\mathrm{holim}_i H(A_i)$ is an $S[[G]]$-module, with

$$\pi_*((\operatorname*{holim}_i H(A_i))_{hG}) \cong H_*^c(G, \lim_i A_i),$$

where $H(A_i)$ is an Eilenberg-Mac Lane spectrum.

In Section 7, we consider an example that arises in chromatic homotopy theory. Let ℓ be a prime and let $n \geq 1$. Let E_n denote the Lubin-Tate spectrum, so that $\pi_*(E_n) = W(\mathbb{F}_{\ell^n})[[u_1, ..., u_{n-1}]][u^{\pm 1}]$, where the degree of u is -2 and the power series ring over the Witt vectors has degree zero. Also, let $G_n = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{\ell^n}/\mathbb{F}_\ell)$ be the extended Morava stabilizer group, where S_n is the nth Morava stabilizer group, and recall that the profinite group G_n is compact ℓ-adic analytic. By work of Paul Goerss, Mike Hopkins, and Haynes Miller (see [18], [10], and [6]), the group G_n acts on E_n.

Now let $K(n)$ be the nth Morava K-theory, with $K(n)_* = \mathbb{F}_\ell[v_n^{\pm 1}]$, where the degree of v_n is $2(\ell^n - 1)$. Also, let $L_{K(n)}(-)$ denote the Bousfield localization functor with respect to $K(n)$. Then we show that the $K(n)$-local Spanier-Whitehead dual of E_n, the function spectrum $F(E_n, L_{K(n)}(S^0))$, is an $S[[G_n]]$-module and there is a homotopy orbit spectral sequence

$$H^c_p(G_n, \pi_q(F(E_n, L_{K(n)}(S^0)))) \Rightarrow \pi_{p+q}(F(E_n, L_{K(n)}(S^0))_{hG_n}).$$

When $(\ell - 1) \nmid n$, the use of the above spectral sequence to identify the spectrum $F(E_n, L_{K(n)}(S^0))_{hG_n}$ more concretely is work in progress.

We want to point out that others have investigated homotopy orbits for profinite groups. In [8], Halvard Fausk studies homotopy orbits in the setting of pro-orthogonal G-spectra, where G is profinite. As pointed out in various places in this paper, Mark Behrens has worked on homotopy orbits for a profinite group. Also, Dan Isaksen has thought about homotopy orbits for profinite groups in the context of pro-spectra. Finally, [12], by Mike Hopkins and Hal Sadofsky, contains some explorations of homotopy orbits involving E_n and G_n. It was the author's study of [12] that motivated him to begin work on the project represented by this paper.

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2. Homotopy orbits when G is finite

In this section, let G be a finite group and let X be a (left) G-spectrum. We use simplicial replacement to obtain an alternative model for the homotopy orbit spectrum X_{hG}, which will be useful for defining the homotopy orbit spectrum when G is a profinite group. Given a spectrum Z, we let Z_k denote the kth pointed simplicial set of Z and we use $Z_{k,l}$ to signify the set of l-simplices of Z_k.

By [2, XII, 5.2],

$$(X_{hG})_k = \mathrm{hocolim}_G X_k = \mathrm{diag}(\textstyle\coprod_* (G \to X_k)),$$

where $\mathrm{diag}(-)$ is the functor that takes the diagonal of a bisimplicial set, and $G \to X_k$ is the diagram out of the groupoid G that is defined by the action of G on X_k. The simplicial replacement is defined as follows:

$$\textstyle\coprod_l (G \to X_k) = \bigvee_{u_l \in I_l} X_k,$$

where I_l is the indexing set that consists of all compositions of the form

$$u_l = i_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_l} i_l,$$

such that each α_j is a morphism in the groupoid G.

The face and degeneracy maps of the pointed simplicial set $\coprod_l (G \to X_k)$ are defined in the following way. Let $0 \leq j < l$ and let $d_j(u_l)$ be the map that, given a morphism $u_l \in I_l$, is defined by the composition

$$X_k \xrightarrow{\mathrm{id}} X_k \longrightarrow \bigvee_{I_{l-1}} X_k,$$

where the target of id is the copy of X_k that is indexed by the morphism

$$u_{l-1} = i_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_l} i_l \in I_{l-1}.$$

Then the jth face map $d_j \colon \bigvee_{I_l} X_k \to \bigvee_{I_{l-1}} X_k$ is the unique map that is induced by all the $d_j(u_l)$, where $d_j(u_l)$ is the canonical map out of the copy of X_k that is indexed by u_l. Also, let $d_l(u_l)$ be the map that, given a morphism $u_l \in I_l$, is defined by the composition

$$X_k \xrightarrow{\alpha_l} X_k \longrightarrow \bigvee_{I_{l-1}} X_k,$$

where the target of α_l is the copy of X_k that is indexed by the morphism

$$u_{l-1} = i_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_{l-1}} i_{l-1} \in I_{l-1}.$$

Then the lth face map $d_l \colon \bigvee_{I_l} X_k \to \bigvee_{I_{l-1}} X_k$ is the unique map that is induced by all the $d_l(u_l)$, where $d_l(u_l)$ is the canonical map out of the copy of X_k that is indexed by u_l.

For $0 \leq j \leq l$, let $s_j(u_l)$ be the map that, given a morphism $u_l \in I_l$, is defined by the composition

$$X_k \xrightarrow{\mathrm{id}} X_k \longrightarrow \bigvee_{I_{l+1}} X_k,$$

where the target of id is the copy of X_k that is indexed by the morphism

$$u_{l+1} = i_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_l} i_l \xleftarrow{\mathrm{id}} i_l \in I_{l+1}.$$

Then the jth degeneracy $s_j \colon \bigvee_{I_l} X_k \to \bigvee_{I_{l+1}} X_k$ is the unique map that is induced by all the $s_j(u_l)$, where $s_j(u_l)$ is the canonical map out of the copy of X_k that is indexed by u_l.

Definition 2.1. Let K be a pointed simplicial set and let L be a set. Then

$$K[L] = K \wedge L_+,$$

where L_+ is the constant simplicial set on L, together with a disjoint basepoint. Similarly, if Z is a spectrum, then $Z[L]$ is the spectrum with each $(Z[L])_k$ equal to $Z_k \wedge L_+$.

Since each u_l can be identified with an element of G^l, the l-fold product of G, $I_l = G^l$ and

$$\textstyle\coprod_l (G \to X_k) = \bigvee_{G^l} X_k \cong X_k \wedge (G^l)_+ = X_k[G^l].$$

Thus,

$$(X_{hG})_k \cong \mathrm{diag}(X_k[G^\bullet]),$$

where the l-simplices of the simplicial pointed simplicial set $X_k[G^\bullet]$ are the pointed simplicial set $X_k[G^l]$.

We introduce some notation that organizes the above, allowing us to summarize it in a theorem.

Definition 2.2. Let X be a G-spectrum. Then $X[G^\bullet]$ is the simplicial spectrum that is defined above, with $(X[G^\bullet])_k = X_k[G^\bullet]$ and l-simplices equal to the spectrum $X[G^l]$. Thus, $X[G^\bullet]$ is the simplicial spectrum

$$X \cong X[*] \leftleftarrows X[G] \Lleftarrow X[G^2] \cdots .$$

We let $\pi_q(X[G^\bullet])$ denote the simplicial abelian group associated to $X[G^\bullet]$.

Definition 2.3. Let $d(Z_\bullet)$ denote the spectrum that is the diagonal of the simplicial spectrum $Z_\bullet$ (see [14, pg. 100]). Thus, for all $k \geq 0$,

$$(d(Z_\bullet))_k = \mathrm{diag}((Z_\bullet)_k)$$

and, for all $l \geq 0$,

$$(d(Z_\bullet))_{k,l} = (Z_l)_{k,l}.$$

Theorem 2.4. *If G is a finite group and X is a G-spectrum, then*

$$X_{hG} \cong d(X[G^\bullet]).$$

3. The homotopy orbit spectrum X_{hG}

In this section, we use the model for the homotopy orbit spectrum that is given in Theorem 2.4 to help us define homotopy orbits for G a profinite group. We begin with a comment about Definition 1.1.

Remark 3.1. Let $X = \mathrm{holim}_i X_i$ be an $S[[G]]$-module. The choice of the term "$S[[G]]$-module" is motivated by the fact that, for each i, the action of G/N_i on X_i yields a function $G/N_i \to \mathrm{Hom}_{\mathrm{Spt}}(X_i, X_i)$ in

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sets}}(G/N_i, \mathrm{Hom}_{\mathrm{Spt}}(X_i, X_i)) &\cong \textstyle\prod_{G/N_i} \mathrm{Hom}_{\mathrm{Spt}}(X_i, X_i)\\ &\cong \mathrm{Hom}_{\mathrm{Spt}}(X_i[G/N_i], X_i)\end{aligned}$$

that corresponds to a map $X_i \wedge (G/N_i)_+ \to X_i$. The analogy with actual modules over the spectrum $S[[G]]$ could be pursued further (as Mark Behrens has done); however, for simplicity, we do not do this.

Using the author's preliminary work on homotopy orbits as a reference point, the following definition was basically given by Behrens, and then later independently formulated by the author.

Definition 3.2. Let $X = \mathrm{holim}_i X_i$ be an $S[[G]]$-module. For each $l \geq 0$, the diagrams $\{X_i\}$ and $\{G/N_i\}$ induce the diagram $\{X_i[(G/N_i)^l]\}$, so that one can form $\mathrm{holim}_i(X_i[(G/N_i)^l])_{\mathtt{f}}$, which gives the simplicial spectrum $\mathrm{holim}_i(X_i[(G/N_i)^\bullet])_{\mathtt{f}}$. Then we define X_{hG}, the *homotopy orbit spectrum* of X with respect to the G-action, to be the spectrum

$$X_{hG} = d(\underset{i}{\mathrm{holim}}(X_i[(G/N_i)^\bullet])_{\mathtt{f}}).$$

Henceforth, we use the notation $X_{h'G}$ to denote the classical homotopy orbit spectrum $\mathrm{hocolim}_G X$, when G is finite.

The definition of the homotopy orbit spectrum comes from imitating the model in Theorem 2.4 and from the demands of the homotopy orbit spectral sequence (see the proof of Theorem 5.3). For example, the reader might expect that the definition involve a homotopy colimit of G-spectra, instead of the homotopy limit of such; however, the homotopy limit is necessitated by the nature of continuous group homology, which is the E_2-term of the homotopy orbit spectral sequence.

Remark 3.3. The functor $(-)_{\mathtt{f}}$ appears in Definition 3.2 so that the homotopy limit is well-behaved. Let us suppose that the functor $(-)_{\mathtt{f}}$ does not appear in the definition of X_{hG}, so that it reads instead as $X_{hG} = d(\operatorname{holim}_i X_i[(G/N_i)^\bullet])$. Recall that $X_i[(G/N_i)^\bullet]$ is involved in the construction of $(X_i)_{hG/N_i} \cong d(X_i[(G/N_i)^\bullet])$, by Theorem 2.4. Thus, one might wonder if there is an isomorphism

$$d(\operatorname*{holim}_i X_i[(G/N_i)^\bullet]) \cong \operatorname*{holim}_i d(X_i[(G/N_i)^\bullet]),$$

so that, morally, $X_{hG} = \operatorname{holim}_i (X_i)_{hG/N_i}$. However, it is not hard to see that this is not the case, since

$$\begin{aligned}(d(\operatorname*{holim}_i X_i[(G/N_i)^\bullet]))_{k,l} &= (\operatorname{diag}(\operatorname*{holim}_i (X_i)_k[(G/N_i)^\bullet]))_l \\ &= (\operatorname*{holim}_i (X_i)_k[(G/N_i)^l])_l \\ &= \operatorname{Hom}_{\mathcal{S}(\{i\}^{\mathrm{op}})}(\Delta[l] \times (\{i\}^{\mathrm{op}})/-, \{(X_i)_k[(G/N_i)^l]\}_i)\end{aligned}$$

(see [2, Chapter XI, 2.2, 3.2]), where $\mathcal{S}$ is the category of simplicial sets, is not equivalent to

$$\begin{aligned}(\operatorname*{holim}_i d(X_i[(G/N_i)^\bullet]))_{k,l} &= (\operatorname*{holim}_i \operatorname{diag}((X_i)_k[(G/N_i)^\bullet]))_l \\ &= \operatorname{Hom}_{\mathcal{S}(\{i\}^{\mathrm{op}})}(\Delta[l] \times (\{i\}^{\mathrm{op}})/-, \{\operatorname{diag}((X_i)_k[(G/N_i)^\bullet])\}_i).\end{aligned}$$

We recall the following useful result.

Theorem 3.4 ([14, Corollary 4.22])**.** *If $X_\bullet$ is a simplicial spectrum, then there is a spectral sequence*

$$E_2^{p,q} = H_p(\pi_q(X_*)) \Rightarrow \pi_{p+q}(d(X_\bullet)),$$

where $H_*(\pi_q(X_*))$ is the homology of the Moore complex of the simplicial abelian group $\pi_q(X_\bullet)$.

Lemma 3.5. *Let $X_\bullet \to Y_\bullet$ be a map between simplicial spectra, such that, for each $n \geq 0$, the map $X_n \to Y_n$ is a weak equivalence between the n-simplices. Then the induced map $d(X_\bullet) \to d(Y_\bullet)$ is a weak equivalence of spectra.*

Proof. There is a spectral sequence

$$H_p(\pi_q(X_*)) \Rightarrow \pi_{p+q}(d(X_\bullet)),$$

and a map to the spectral sequence

$$H_p(\pi_q(Y_*)) \Rightarrow \pi_{p+q}(d(Y_\bullet)).$$

Since $\pi_q(X_n) \cong \pi_q(Y_n)$, for each $n \geq 0$, and $\pi_q(X_*)$ and $\pi_q(Y_*)$ are chain complexes, there is an isomorphism $H_p(\pi_q(X_*)) \xrightarrow{\cong} H_p(\pi_q(Y_*))$ of E_2-terms. Therefore, the abutments of the above two spectral sequences are isomorphic, giving the conclusion of the lemma. $\square$

Remark 3.6. Let G be finite and let X be any G-spectrum. Let $\{i\} = \{0\}$, $N_0 = \{e\}$, and $X_0 = X_{\mathtt{f}}$. Then $\operatorname{holim}_{\{0\}} X_0 \cong X_{\mathtt{f}}$ is an $S[[G]]$-module and

$$(\operatorname*{holim}_{\{0\}} X_0)_{hG} = d(\operatorname*{holim}_{\{0\}} (X_{\mathtt{f}}[G^\bullet])_{\mathtt{f}}) \cong d((X_{\mathtt{f}}[G^\bullet])_{\mathtt{f}}).$$

Thus, there is a weak equivalence

$$X_{h'G} \cong d(X[G^\bullet]) \xrightarrow{\simeq} d((X_{\mathtt{f}}[G^\bullet])_{\mathtt{f}}) \cong (\operatorname*{holim}_{\{0\}} X_0)_{hG},$$

so that Definition 3.2 recovers the classical definition of homotopy orbits.

Now we show that $(-)_{hG}$ preserves weak equivalences of $S[[G]]$-modules, as defined below.

Definition 3.7. Let $\{X_i\} \to \{Y_i\}$ be a natural transformation of diagrams of G-spectra, such that $X_i \to Y_i$ is a weak equivalence, for each i, and the induced map $X = \operatorname{holim}_i X_i \to \operatorname{holim}_i Y_i = Y$ is a map between $S[[G]]$-modules. Then we say that the weak equivalence $X \to Y$ is a *weak equivalence of $S[[G]]$-modules.*

Theorem 3.8. *If $X \to Y$, as in Definition 3.7, is a weak equivalence of $S[[G]]$-modules, then $X_{hG} \to Y_{hG}$ is a weak equivalence.*

Proof. Let $i \geq 0$. Since $X_i \to Y_i$ is a weak equivalence, the induced map

$$(X_i[(G/N_i)^l])_{\mathtt{f}} \to (Y_i[(G/N_i)^l])_{\mathtt{f}}$$

is a weak equivalence between fibrant spectra, for each $l \geq 0$. Thus,

$$\operatorname*{holim}_i (X_i[(G/N_i)^l])_{\mathtt{f}} \to \operatorname*{holim}_i (Y_i[(G/N_i)^l])_{\mathtt{f}}$$

is a weak equivalence, so that, by Lemma 3.5, $X_{hG} \to Y_{hG}$ is a weak equivalence. □

4. Examples of $S[[G]]$-modules

In this section, we consider a way that $S[[G]]$-modules arise naturally from discrete right G-spectra.

The following recollection will be helpful. There is a functor $(-)_c \colon \mathrm{Spt} \to \mathrm{Spt}$, such that, given Y in Spt, Y_c is a cofibrant spectrum, and there is a natural transformation $(-)_c \to \mathrm{id}_{\mathrm{Spt}}$, such that, for any Y, the map $Y_c \to Y$ is a trivial fibration. For example, if Y is a right K-spectrum for some group K, then Y_c is also a right K-spectrum, and the map $Y_c \to Y$ is K-equivariant.

Let Z be any spectrum and let X be a discrete right G-spectrum. Then

$$F(X, Z) \simeq F(\operatorname*{colim}_i X^{N_i}, Z_{\mathtt{f}}) \simeq F(\operatorname*{hocolim}_i X^{N_i}, Z_{\mathtt{f}}) \simeq \operatorname*{holim}_i F((X^{N_i})_c, Z_{\mathtt{f}}).$$

Thus, when X is a discrete right G-spectrum, we identify the left G-spectrum $F(X, Z)$ with the G-spectrum $\operatorname{holim}_i F((X^{N_i})_c, Z_{\mathtt{f}})$. Under this identification, we make the following observation.

Theorem 4.1. *If X is a discrete right G-spectrum, then $F(X, Z)$ is an $S[[G]]$-module.*

Proof. The spectrum $(X^{N_i})_c$ is a right G/N_i-spectrum, because X^{N_i} is a right G/N_i-spectrum. Thus, $F((X^{N_i})_c, Z_{\mathtt{f}})$ is a G/N_i-spectrum. Also, since the source and target of the function spectrum $F((X^{N_i})_c, Z_{\mathtt{f}})$ are cofibrant and fibrant, respectively, the function spectrum itself is fibrant. These facts imply that the spectrum $\operatorname{holim}_i F((X^{N_i})_c, Z_{\mathtt{f}})$ is an $S[[G]]$-module. □

We give two examples of $F(X, Z)$ as an $S[[G]]$-module.

Example 4.2. Recall from [3] that, if cS^0 is the Brown-Comenetz dual of S^0, then cX, the Brown-Comenetz dual of X, is $F(X, cS^0)$. Thus, if X is a discrete right G-spectrum, then

$$cX = \operatorname*{holim}_i F((X^{N_i})_c, (cS^0)_{\mathtt{f}})$$

is an $S[[G]]$-module.

The following example is due to Mark Behrens.

Example 4.3. If X is a discrete right G-spectrum, then the Spanier-Whitehead dual of X,

$$DX = F(X, S^0) = \operatorname{holim}_i F((X^{N_i})_c, (S^0)_{\mathtt{f}}),$$

is an $S[[G]]$-module.

5. The homotopy orbit spectral sequence for an $S[[G]]$-module

In this section, we construct a homotopy orbit spectral sequence for a certain type of $S[[G]]$-module. We attempted to construct such a spectral sequence for any profinite group G. However, there were difficulties that we were able to get around only by putting an additional hypothesis on the group G, that is, by assuming that G is countably based.

Now we recall the construction of the classical homotopy orbit spectral sequence for $X_{h'G}$; the brief analysis below of its E_2-term will be useful. Thus, for the duration of this paragraph, let G be a finite group and let X be a G-spectrum. By Theorem 3.4, there is a spectral sequence

$$E_2^{p,q} \Rightarrow \pi_{p+q}(d(X[G^\bullet])) = \pi_{p+q}(X_{h'G}),$$

where

$$E_2^{p,q} = H_p(\pi_q(X[G^*])) \cong H_p(G, \pi_q(X)), \tag{5.1}$$

the pth group homology of G, with coefficients in $\pi_q(X)$ (see, for example, [14, (7.9)]). Note that, by (5.1),

$$H_p(G, \pi_q(X)) \cong H_p(\pi_q(X)[G^*]), \tag{5.2}$$

where, if A is an abelian group and K is a group, we use $A[K]$ to denote $\bigoplus_K A$.

Now we are ready to construct the homotopy orbit spectral sequence for a countably based profinite group.

Theorem 5.3. *Let G be a countably based profinite group, and let*

$$N_0 \geq N_1 \geq N_2 \geq \cdots$$

be a chain of open normal subgroups of G, such that $G \cong \lim_i G/N_i$. Also, let

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

be a tower of G-spectra and G-equivariant maps such that $X = \operatorname{holim}_i X_i$ is an $S[[G]]$-module. If, for all i, X_i is an f-spectrum, then there is a spectral sequence of the form

$$E_2^{p,q} \cong H_p^c(G, \pi_q(X)) \Rightarrow \pi_{p+q}(X_{hG}),$$

where the E_2-term is the continuous homology of G with coefficients in the graded profinite $\widehat{\mathbb{Z}}[[G]]$-module $\pi_*(X)$.

Proof. By Theorem 3.4, there is a spectral sequence

$$E_2^{p,q} \Rightarrow \pi_{p+q}(d(\operatorname{holim}_i(X_i[(G/N_i)^\bullet])_{\mathtt{f}})) = \pi_{p+q}(X_{hG}),$$

where

$$E_2^{p,q} = H_p(\pi_q(\operatorname{holim}_i(X_i[(G/N_i)^*])_{\mathtt{f}})).$$

We spend the rest of the proof identifying the E_2-term as continuous group homology.

Let $l \geq 0$. Since $\{\pi_{q+1}(X_i)[(G/N_i)^l]\}$ is a tower of finite abelian groups,

$$\lim_i{}^1 \pi_{q+1}(X_i)[(G/N_i)^l] = 0.$$

Thus,

$$\pi_q(\operatorname{holim}_i (X_i[(G/N_i)^l])_{\mathtt{f}}) \cong \lim_i \pi_q(X_i)[(G/N_i)^l],$$

which implies that

$$E_2^{p,q} \cong H_p(\lim_i(\pi_q(X_i)[(G/N_i)^*])).$$

Since, for any $l \geq 0$, the tower of abelian groups $\{\pi_q(X_i)[(G/N_i)^l]\}_i$ satisfies the Mittag-Leffler condition, by [23, Theorem 3.5.8], there is a short exact sequence

$$0 \to \lim_i{}^1 H_{p+1}(\pi_q(X_i)[(G/N_i)^*]) \to E_2^{p,q} \to \lim_i H_p(\pi_q(X_i)[(G/N_i)^*]) \to 0.$$

By (5.2), this short exact sequence can be written as

$$0 \to \lim_i{}^1 H_{p+1}(G/N_i, \pi_q(X_i)) \to E_2^{p,q} \to \lim_i H_p(G/N_i, \pi_q(X_i)) \to 0.$$

Since G/N_i and $\pi_q(X_i)$ are finite, $H_{p+1}(G/N_i, \pi_q(X_i))$ is a finite abelian group, for each $p \geq 0$ ([23, Corollary 6.5.10]). Thus, $\lim^1{}_i H_{p+1}(G/N_i, \pi_q(X_i)) = 0$. Therefore, we obtain that

$$E_2^{p,q} \cong \lim_i H_p(G/N_i, \pi_q(X_i)) \cong H_p^c(G, \pi_q(X)),$$

where the last isomorphism uses [19, Proposition 6.5.7]. □

6. Eilenberg-Mac Lane spectra and their homotopy orbits

Let the profinite group G be countably based, with a chain $\{N_i\}$ of open normal subgroups, such that $G \cong \lim_i G/N_i$, and, as defined in the Introduction, let $\{A_i\}$ be a nice tower of G-modules. Let $\Gamma\colon \mathbf{Ch}_+ \to s\mathbf{Ab}$ be the functor in the Dold-Kan correspondence from $\mathbf{Ch}_+$, the category of chain complexes C_* with $C_n = 0$ for $n < 0$, to $s\mathbf{Ab}$, the category of simplicial abelian groups (see, for example, [11, Chapter III, Corollary 2.3]). Also, if A is an abelian group, let $A[-n]$ be the chain complex that is A in degree n and zero elsewhere.

Given the tower $\{A_i\}$, we explain how to form $\{H(A_i)\}$, a tower of Eilenberg-Mac Lane spectra (we follow the construction given in [15]), so that $\operatorname{holim}_i H(A_i)$ is an $S[[G]]$-module. By functoriality, for each $k \geq 0$, $\{\Gamma(A_i[-k])\}$ is a tower of simplicial G-modules and G-equivariant maps, such that, for each i, $\Gamma(A_i[-k])$ is the Eilenberg-Mac Lane space $K(A_i, k)$ and $\Gamma(A_i[-k])$ is a simplicial G/N_i-module. Furthermore, by taking 0 as the basepoint, each $\Gamma(A_i[-k])$ is a pointed simplicial set.

For each i, we define the Eilenberg-Mac Lane spectrum $H(A_i)$ by $(H(A_i))_k = \Gamma(A_i[-k])$, so that $\pi_0(H(A_i)) = A_i$ and $\pi_n(H(A_i)) = 0$, when $n \neq 0$. Then, by functoriality, $\{H(A_i)\}$ is a tower of G-spectra and G-equivariant maps, such that each $H(A_i)$ is a G/N_i-spectrum. Since each $(H(A_i))_k$ is a fibrant simplicial set and each $H(A_i)$ is an Ω-spectrum (see, for example, [15, Example 21]), $H(A_i)$ is a fibrant spectrum. These facts imply the following result.

Lemma 6.1. *The spectrum* $\operatorname{holim}_i H(A_i)$ *is an* $S[[G]]$*-module.*

Now we show that the homotopy orbit spectral sequence can be used to compute $\pi_*((\operatorname{holim}_i H(A_i))_{hG})$.

Theorem 6.2. *If G is a countably based profinite group with a chain $\{N_i\}$ of open normal subgroups, such that $G \cong \lim_i G/N_i$, and, if $\{A_i\}$ is a nice tower of G-modules, then*

$$\pi_*((\operatorname{holim}_i H(A_i))_{hG}) \cong H_*^c(G, \lim_i A_i).$$

Proof. By hypothesis, each $H(A_i)$ is an f-spectrum. Then, by Theorem 5.3, there is a homotopy orbit spectral sequence

$$E_2^{p,q} \cong H_p^c(G, \pi_q(\operatorname{holim}_i H(A_i))) \Rightarrow \pi_{p+q}((\operatorname{holim}_i H(A_i))_{hG}).$$

Since $\lim^1{}_i A_i = 0$, we have:

$$E_2^{p,q} \cong H_p^c(G, \lim_i \pi_q(H(A_i))) = \begin{cases} H_p^c(G, 0) & \text{if } q \neq 0 \\ H_p^c(G, \lim_i A_i) & \text{if } q = 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } q \neq 0 \\ H_p^c(G, \lim_i A_i) & \text{if } q = 0. \end{cases}$$

Thus, the spectral sequence collapses, giving the conclusion of the theorem. □

7. The G_n-homotopy orbits of $F(E_n, L_{K(n)}(S^0))$

Our main motivation for constructing homotopy orbits for a profinite group was that we were interested in constructing the G_n-homotopy orbit spectrum of E_n, as part of our effort to understand [12]. Though we have not been able to construct such a spectrum, we are able to construct the G_n-homotopy orbits of the closely related spectrum

$$F(E_n, L_{K(n)}(S^0)) \simeq \Sigma^{-n^2} E_n,$$

where the weak equivalence applies [20, Proposition 16] and the function spectrum $F(E_n, L_{K(n)}(S^0))$ is the $K(n)$-local Spanier-Whitehead dual of E_n. After explaining why we have not been able to construct a homotopy orbit spectrum for E_n, we consider $F(E_n, L_{K(n)}(S^0))_{hG_n}$.

Recall from the Introduction that, since G_n is a compact ℓ-adic analytic group, it is a countably based profinite group. Then, following [6, (1.4)], we fix a chain $G_n = N_0 \gneqq N_1 \gneqq N_2 \gneqq \cdots$ of open normal subgroups of G_n, such that $\bigcap_i N_i = \{e\}$, so that $G_n \cong \lim_i G_n/N_i$. Thus, an $S[[G_n]]$-module comes from a tower $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ of G_n-spectra and G_n-equivariant maps.

To form the G_n-homotopy orbit spectrum of E_n, we have to show that E_n is an $S[[G_n]]$-module. However, we do not know how to do this and we explain how a natural way to try to do this fails. Let $E_n^{dhN_i}$ be the spectrum constructed by Devinatz and Hopkins (see [6]) that behaves like the N_i-homotopy fixed points of E_n with respect to a continuous N_i-action (see [4]). Let

$$M_{I_0} \leftarrow M_{I_1} \leftarrow M_{I_2} \leftarrow \cdots$$

be a tower of generalized Moore spectra such that

$$\operatorname{holim}_j (L_{K(n)}(M_{I_j}))_{\mathtt{f}} \simeq L_{K(n)}(S^0)$$

([13, Proposition 7.10 (e)]). Also, recall from [4, Theorem 6.6] that

$$E_n \simeq \operatorname{holim}_j (\operatorname{colim}_i (E_n^{dhN_i} \wedge M_{I_j})_{\mathtt{f}}).$$

Thus, one might attempt to present E_n as an $S[[G_n]]$-module by considering the tower $\{\mathrm{colim}_i(E_n^{dhN_i} \wedge M_{I_j})_{\mathtt{f}}\}_j$. However, by [4, Lemma 6.2], there is no open normal subgroup N of G_n such that the G_n-action on $\mathrm{colim}_i(E_n^{dhN_i} \wedge M_{I_j})_{\mathtt{f}}$ factors through G_n/N. Thus, this attempt fails to present E_n as an $S[[G_n]]$-module.

We noted above that $F(E_n, L_{K(n)}(S^0)) \simeq \Sigma^{-n^2} E_n$. Thus, one might like to assert that, since we can construct the G_n-homotopy orbit spectrum of $F(E_n, L_{K(n)}(S^0))$, as done below, then we can also construct the homotopy orbit spectrum of E_n, by considering the homotopy orbits of $\Sigma^{n^2} F(E_n, L_{K(n)}(S^0))$. However, we have been informed by Mark Behrens that, in general, this can not be done, because the equivalence $F(E_n, L_{K(n)}(S^0)) \simeq \Sigma^{-n^2} E_n$ need not be G_n-equivariant. According to Behrens, the S^{-n^2} in $\Sigma^{-n^2} E_n$, when properly interpreted, can have a non-trivial G_n-action on it.

To form $F(E_n, L_{K(n)}(S^0))_{hG_n}$, we only need to show that $F(E_n, L_{K(n)}(S^0))$ is an $S[[G_n]]$-module. As in [4], let $F_n = \mathrm{colim}_i\, E_n^{dhN_i}$. Then

$$\begin{aligned}
F(E_n, L_{K(n)}(S^0)) &\simeq F(L_{K(n)}(F_n), \operatorname*{holim}_j (L_{K(n)}(M_{I_j}))_{\mathtt{f}}) \\
&\simeq F(F_n, \operatorname*{holim}_j (L_{K(n)}(M_{I_j}))_{\mathtt{f}}) \\
&\simeq F(\operatorname*{hocolim}_i (E_n^{dhN_i})_c, \operatorname*{holim}_j (L_{K(n)}(M_{I_j}))_{\mathtt{f}}) \\
&\cong \operatorname*{holim}_i \operatorname*{holim}_j F((E_n^{dhN_i})_c, (L_{K(n)}(M_{I_j}))_{\mathtt{f}}) \\
&\simeq \operatorname*{holim}_i F((E_n^{dhN_i})_c, (L_{K(n)}(M_{I_i}))_{\mathtt{f}}),
\end{aligned}$$

where the first weak equivalence applies [6, Definition 1.5] and [4, Theorem 6.3], and the last weak equivalence uses the fact that $\{(i,i)\}_i$ is cofinal in $\{(i,j)\}_{i,j}$. Thus, we make the identification

$$F(E_n, L_{K(n)}(S^0)) = \operatorname*{holim}_i F((E_n^{dhN_i})_c, (L_{K(n)}(M_{I_i}))_{\mathtt{f}}).$$

Since $E_n^{dhN_i}$ is a right G_n/N_i-spectrum, as in Section 4, the above identification implies the following result.

Lemma 7.1. *The spectrum $F(E_n, L_{K(n)}(S^0))$ is an $S[[G_n]]$-module.*

The next result allows us to build the homotopy orbit spectral sequence for $F(E_n, L_{K(n)}(S^0))_{hG_n}$.

Lemma 7.2. *For each $i \geq 0$, the spectrum $F(E_n^{dhN_i}, L_{K(n)}(M_{I_i}))$ is an f-spectrum.*

Proof. Our proof follows [5, proof of Lemma 3.5]. Since

$$L_{K(n)}(M_{I_i}) \simeq L_{K(n)}(S^0) \wedge M_{I_i} \simeq E_n^{dhG_n} \wedge M_{I_i}$$

(see [6, Theorem 1(iii)]), [6, Theorem 2(ii)] gives a strongly convergent $K(n)$-local E_n-Adams spectral sequence

$$H_c^s(G_n, (E_n \wedge M_{I_i})^{-t}(E_n^{dhN_i})) \Rightarrow \pi_{t-s}(F(E_n^{dhN_i}, L_{K(n)}(M_{I_i}))),$$

where the E_2-term is continuous cohomology for profinite continuous $\mathbb{Z}_\ell[[G_n]]$-modules (see [6, Remark 1.3]). By [9, Proposition 2.5],

$$(E_n \wedge M_{I_i})^{-t}(E_n^{dhN_i}) \cong \textstyle\bigoplus_{G_n/N_i} \pi_t(E_n \wedge M_{I_i})$$

is a finite abelian group. Thus, by [21, Proposition 4.2.2], the abelian group $H^s_c(G_n, (E_n \wedge M_{I_i})^{-t}(E_n^{dhN_i}))$ is finite. The fact that $E_\infty^{*,*}$ has a horizontal vanishing line (see [1, Theorem 6.10], [5, Proposition 2.3, proof of Lemma 3.5], and [6, Proposition A.3]) implies that $\pi_*(F(E_n^{dhN_i}, L_{K(n)}(M_{I_i})))$ is finite in each degree. □

Theorem 5.3 and Lemmas 7.1 and 7.2 give the following.

Theorem 7.3. *There is a spectral sequence*

$$H^c_p(G_n, \pi_q(F(E_n, L_{K(n)}(S^0)))) \Rightarrow \pi_{p+q}(F(E_n, L_{K(n)}(S^0))_{hG_n}).$$

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